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A review on the research of mechanical problems with different moduli in tension and compression †

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Abstract

Most materials exhibit different tensile and compressive strains given the same stress applied in tension or compression. These materials are known as bimodular materials. An important model of bimodular materials is the criterion of positive-negative signs of principal stress proposed by Ambartsumyan. This model is mainly applicable to isotropic materials and deals with the principal stress state in a point. However, due to the inherent complexity of the constitutive relation, FEM based on iterative strategy and analytical methods based on a simplified mechanical model are required. In this paper, we review the basic assumptions of this model and its development, several innovative computational methods, and some important engineering applications. We also discuss the sequent key problems in this field.

Keywords: Bimodulus; Tension and compression; Constitutive model; FEM; Analytical solution

1. Introduction

Classical elasticity theory assumes that materials have the same elastic properties in tension and compression, but this is only a simplified interpretation, and does not account for material nonlinearities. Many studies have indicated that most materials, including concrete, ceramics, graphite, and some composites, exhibit different tensile and compressive strains given the same stress applied in tension or compression. Thus, materials exhibit different elastic moduli in tension and compression. These materials are known as bimodular materials [1-3]. Overall, there are two basic material models widely used in theoretical analysis within the engineering profession. One of these models is the criterion of positive-negative signs in the longitudinal strain of fibers proposed by Bert [4]. This model is mainly applicable to orthotropic materials, and is therefore widely used for research on laminated composites [5-10]. Another model is the criterion of positive-negative signs of principal stress proposed by Ambartsumyan [11]. This model is mainly applicable to isotropic materials and the earlier study is only seen in Kamiya's work [12-14]. In mechanical engineering, the stress state in a principal direction is a key point in the analysis of some components like beams, columns, plates and shells. However, shear stresses and the resulting diagonal tension in principal direction must also be carefully considered in the design of structures. The discussions in this paper will be focused on the latter model based on principal direction.

The elasticity theory of different moduli in tension and compression presented by Ambartsumyan [11] asserts that Young's modulus depends not only on material properties, but also on the stress state of the point in question. Therefore, the elastic modulus is related to the material, shape, boundary conditions, and external loads of the structure, and hence has nonlinear characteristics. Due to the materials nonlinearity, FEM based on iterative strategy and analytical methods based on a simplified mechanical model are required. In this paper, we will focus our discussion on the new progresses in numerical iteration and analytical model, and their applications in the engineering structures.

2. Basic theory and development

2.1 Basic theory

Ambartsumyan linearized the nonlinear model, the second material model mentioned above, into two straight lines whose tangents at the origin are discontinuous, as shown in Fig. 1.

This bimodular model follows the common rules of elastic continuum mechanics. The basic assumptions of this theory are as follows:

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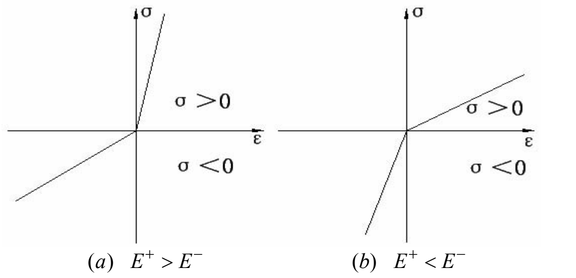

Fig. 1. Constitutive model of bimodular materials.

(1) The studied body is continuous, homogeneous and isotropic.

(2) Small deformation is assumed.

(3) Young's modulus of elasticity and Poisson's ratio of materials are E^+ and μ^+, respectively while the materials is in tension along certain direction; and they are E^- and μ^-, respectively while is in compression along certain direction.

(4) When the three principal stresses are uniformly positive or uniformly negative i.e. the first class, the three basic equations are essentially the same as those of classical theory. In the state of general stress, however, the sign of a certain principal stress can be different from the sign of the other two principal stresses, i.e., the second class. When the signs of the three principal stresses are different, the differential equations of equilibrium and the geometrical equations are the same as those of classical materials theory, with the exception of the physical equations.

(5) $\mu^+ / E^+ = \mu^- / E^-$ is introduced and the assumption ensures symmetry of the flexibility matrix.

In a spatial problem, let the stress and strain components in the principal coordinates α, β, γ be, $\{\sigma_I\} = (\sigma_\alpha\ \sigma_\beta\ \sigma_\gamma)^T$ and $\{\varepsilon_I\} = (\varepsilon_\alpha\ \varepsilon_\beta\ \varepsilon_\gamma)^T$, respectively. The constitutive model proposed by Ambartsumyan is

$$\begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\gamma \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\gamma \end{Bmatrix}, \tag{1}$$

where $a_{ij}(i, j = 1,2,3)$ denotes the flexibility coefficients determined by the polarity of the signs of the principal stresses. For instance, if $\sigma_\alpha > 0, \sigma_\beta < 0, \sigma_\gamma > 0$, the physical relation should be

$$\begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\gamma \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{E^+} & -\dfrac{\mu^-}{E^-} & -\dfrac{\mu^+}{E^+} \\ -\dfrac{\mu^+}{E^+} & \dfrac{1}{E^-} & -\dfrac{\mu^+}{E^+} \\ -\dfrac{\mu^+}{E^+} & -\dfrac{\mu^-}{E^-} & \dfrac{1}{E^+} \end{bmatrix} \begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\gamma \end{Bmatrix}. \tag{2}$$

The rest of the physical equations may be deduced analogously. Due to the fact that $\mu^+ / E^+ = \mu^- / E^-$, the symmetry of the flexibility matrix is assured. Therefore, $a_{12} = a_{21} = a_{13} = a_{31} = a_{23} = a_{32}$. Eq. (2) may be rewritten as

$$\begin{cases} \varepsilon_\alpha = a_{11}\sigma_\alpha + a_{12}\sigma_\beta + a_{12}\sigma_\gamma \\ \varepsilon_\beta = a_{12}\sigma_\alpha + a_{22}\sigma_\beta + a_{12}\sigma_\gamma \\ \varepsilon_\gamma = a_{12}\sigma_\alpha + a_{12}\sigma_\beta + a_{33}\sigma_\gamma \end{cases}, \tag{3}$$

where only four flexibility coefficients, $a_{11}, a_{22}, a_{33}, a_{12}$, are independent.

2.2 Development

In a state of complex stress, the determination of the flexibility coefficients in material matrix is difficult because the elastic constants E^+, μ^+ and E^-, μ^- is related to the signs and the directions of the principal stresses, which are dependent upon the external load applied, the structural shape, the boundary conditions and so on. Researchers gave the following modified material models in the form of matrix:

(1) Jones's materials model

In a state of two-dimensional stress, if $\sigma_\alpha > 0$, $\sigma_\beta < 0$, the elasticity constants are [1, 2]

$$a_{11} = \frac{1}{E^+}, a_{22} = \frac{1}{E^-}, a_{12} = a_{21} = -k_\alpha \frac{\mu^+}{E^+} - k_\beta \frac{\mu^-}{E^-}, \tag{4}$$

where

$$k_\alpha = \frac{|\sigma_\alpha|}{|\sigma_\alpha| + |\sigma_\beta|}, k_\beta = \frac{|\sigma_\beta|}{|\sigma_\alpha| + |\sigma_\beta|} \tag{5}$$

and k_α, k_β are the weighted coefficients. The constitutive relationship of Jones's materials model in the form of matrix is under the influence of the signs and the magnitudes of the principal stresses. The weighted coefficients should be determined numerically by the further tests.

(2) Vijayakumar's and Li's materials model

Based on the principal stress, the principal strain and the strain energy in tension and compression, Vijayakumar and Rao [15] proposed that the computational model may be divided into several small submatrixes and the areas in tension and compression may be divided into the less area to analyze. In addition, in the analysis of shell structures with different moduli, Li [16] proposed that the stress-strain relation may be expressed in several small parameters by means of asymptotic expansion.

(3) Ye's materials model

Ye et al. [17, 18] proposed that the elasticity coefficients $C_{il}(i, l = 1,2,3)$ may be selected under the criterion on the signs in the principal strains, the relation between the principal stress and principal strain may be taken as

$$\begin{Bmatrix} \sigma_\alpha \\ \sigma_\beta \\ \sigma_\gamma \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_\alpha \\ \varepsilon_\beta \\ \varepsilon_\gamma \end{Bmatrix}, \tag{6}$$

where the signs of $C_{il}(i, l = 1,2,3)$ are determined by the polarity in the principal strains: if $\varepsilon_\alpha > 0$ then C_{il}^+ is adopted; if $\varepsilon_\alpha < 0$ then C_{il}^- is adopted. The definition based on the principal strain is intuitionistic, and it is easy to test by experiments. So, Ye's criterion on the principal strain is innovative.

3. FEM based on iterative strategy

3.1 Key problem

Zhang and Wang [19] put forward FEM based on different moduli in tension and compression and pointed out that the essential difference between FEM on single modulus and FEM on different moduli lies in the elasticity matrix $[\overline{D}]$. Their work established a theoretical basis for numerical iterative computation based on different moduli in tension and compression.

The total potential energy function of the computing model of finite elements is

$$\Pi_p = \sum_{i=1}^{M}[\frac{1}{2}\int_{v_i}\{\varepsilon\}_i^T[\overline{D}]_i\{\varepsilon\}_i dv - \int_{v_i}\{f\}_i^T\{\overline{F}\}_i dv - \int_{S_{\sigma i}}\{f\}_i^T\{\overline{P}\}_i ds], \tag{7}$$

where M is the total of elements, $\{\varepsilon\}_i$ is the strain matrix of element *i*, $[\overline{D}]_i$ is the elasticity matrix of element *i*, $\{f\}_i$ is the displacement function of element *i*, $\{\overline{F}\}_i$ is the body force matrix of element *i*, $\{\overline{P}\}_i$ is the surface force matrix of element *i*. Under the assumption of $\mu^+/E^+ = \mu^-/E^-$, the governing equation of FEM is

$$[K]\{d\} = \{P\}, \tag{8}$$

where $[K]$ is the global stiffness matrix, $\{d\}$ is the global nodal displacement matrix, $\{P\}$ is the global nodal load matrix.

The bimodular materials model proposed by Ambartsumyan asserts that Young's modulus depends not only on material properties, but also on the stress state of that point. There are only a few applications of the constitutive equation to stress analyses of components because of the inherent complexity in the analysis of the bimodular materials, i.e. the elasticity constants involved in the governing equations, which depend on the stress state of that point, are not correctly indicated beforehand. In other words, except in particularly simple problems it is not easy to estimate the stress state in a point in the deformed body *a prior*. In some complex problems, it is necessary to resort to FEM based on an iterative strategy. Generally, there are two basic approaches: one is the direct iterative method based on a variable stiffness matrix, that is, in the each iteration, the principal stress state are determined again to modify the relevant elasticity matrix for the next iteration; the other is that the stiffness matrix may be triangularized only once. However, the computational effort and the convergent rate of the latter will depend on the selection of initial values and parameters to a great extent. The following is intended to be a brief introduction to the main progresses of the numerical computation.

3.2 Zhang's improved algorithm

Due to the slow convergent rate and the instability in iteration, Zhang and Wang [19] put forward an improved algorithm for accelerating convergence based on the idea of integral matrix. Thereafter, Liu and Zhang [20] and He et al. [21] perfected this improved algorithm in how to determine theoretically the shear modulus of elasticity.

To satisfy the regression and accelerate convergence, Zhang and Wang [19] thought the matrices should have an integral feature. Therefore, shear stress and shear strain are set equal to zero to formulate physical Eq. (1), the principal stress and the principal strain (in a plane problem) may be written as

$$\{\sigma_I\} = [\sigma_\alpha \ \sigma_\beta \ \tau_{\alpha\beta}]^T, \{\varepsilon_I\} = [\varepsilon_\alpha \ \varepsilon_\beta \ \varepsilon_{\alpha\beta}]^T, \tag{9}$$

where $\tau_{\alpha\beta} = \varepsilon_{\alpha\beta} = 0$. The relation between stress and strain in the principal direction and the corresponding elasticity matrix are, respectively

$$\{\sigma_I\} = [D]\{\varepsilon_I\}, \quad [D] = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix}, \tag{10}$$

where d_{33} is essentially the introduced shear modulus of elasticity. When elastic moduli in tension and compression are the same, it may be determined uniquely as

$$d_{33} = \frac{E}{2(1+\mu)}. \tag{11}$$

However, it is very difficult to determine d_{33} when different moduli in tension and compression must be taken into account. As a result, the effective application of this convergent algorithm was further limited by the indeterminacy of item d_{33}.

When the point in question belongs to the first class discussed above, all principal stresses are uniformly positive ($G^+ = E^+/2(1+\mu^+)$) or uniformly negative ($G^- = E^-/2(1+\mu^-)$). The two shear moduli are essentially the same as those of classical theory, and they are thus easily obtained. However, when the point in question belongs to the second class discussed above, the signs of the three principal stresses are different. Because of this, we will inevitably encounter difficulties when determining the shear modulus pattern. In past computations, G has been taken as an average over the tensile-compressive elasticity moduli and the tensile-compressive Poisson's ratios, i.e.

$$G = \frac{(E^+ + E^-)/2}{2[1+(\mu^+ + \mu^-)/2]} = \frac{E^+ + E^-}{2(1+\mu^+) + 2(1+\mu^-)}. \tag{12}$$

Eq. (12) neglects the influences brought about by the stress state of the point in question, so it is not an optimal solution. Based on the idea that G should be weighted according to the ratio of tensile or compressive principal stresses to the sum of all principal stresses in absolute value, Liu and Zhang [20] proposed

$$G_{xy} = \frac{\eta E^+ + (1-\eta)E^-}{2\eta(1+\mu^+) + 2(1-\eta)(1+\mu^-)}, \tag{13}$$

where η is a factor for accelerating convergence, and its value is the ratio of positive principal stress to the sum of the three principal stresses in absolute value, such that $0 \le \eta \le 1$. By multiplying the items E^+ and $2(1+\mu^+)$ by η, and multiplying the items E^- and $2(1+\mu^-)$ by $1-\eta$, we easily obtain Eq. (13) from Eq. (12). Consequently, a strict derivation in theory is necessary in order to eliminate the need for *a priori* assumptions.

Using the general elasticity law from Ambartsumyan's theory, and combining with the idea of accelerated convergence, He et al. [21] lastly derived the exact pattern of the shear modulus of elasticity as follows

$$G_{xy} = \frac{\eta E^+E^- + (1-\eta)E^+E^-}{2\eta(1+\mu^+)E^- + 2(1-\eta)(1+\mu^-)E^+}. \tag{14}$$

If the shear moduli in tension and compression, i.e. $G^+ = E^+/2(1+\mu^+)$ and $G^- = E^-/2(1+\mu^-)$, are introduced in Eq. (14), then the pattern of shear modulus may be rewritten as

$$G_{xy} = \frac{1}{\eta\dfrac{2(1+\mu^+)}{E^+} + (1-\eta)\dfrac{2(1+\mu^-)}{E^-}} = \frac{1}{\dfrac{\eta}{G^+} + \dfrac{1-\eta}{G^-}} \tag{15}$$

It is seen that η can directly act on the shear moduli in tension and compression, G^+ and G^-, therefore, the pattern is concise. Also, the computational examples indicated that this pattern has a better convergence, compared with the effect of the existing formulas.

In addition, Zhang et al. [22] also made an error analysis on the assumption $\mu^+/E^+ = \mu^-/E^-$ and gave the upper limit of errors.

3.3 Ye's improved algorithm

Ye et al. [23, 24] proposed another algorithm more suitable for engineering applications, which the Poisson's ratio keeps constant while modifying the elasticity matrix. Ye et al. gave the following elasticity matrix

$$[D_I] = \begin{bmatrix} \dfrac{E_1}{1-\nu^2} & \dfrac{E_1\nu}{1-\nu^2} \\ \dfrac{E_2\nu}{1-\nu^2} & \dfrac{E_2}{1-\nu^2} \end{bmatrix} or\ [D_I] = \begin{bmatrix} \dfrac{E_1}{1-\nu^2} & \dfrac{E_2\nu}{1-\nu^2} \\ \dfrac{E_1\nu}{1-\nu^2} & \dfrac{E_2}{1-\nu^2} \end{bmatrix}, \tag{16}$$

where ν is the Poisson's ratio, E_1 and E_2 are the elasticity moduli in tension and compression determined by the signs of the principal strains, respectively. The governing equation is

$$\frac{[K]+[K]^T}{2}\{d\} = \{P\}. \tag{17}$$

Besides, Ye et al. modified the stiffness matrix from all the existent models by using the idea on equivalence. Ye's improved algorithm is consistent with the criterion on principal strain i.e., Ye's materials model, which has been introduced in Section 2.2.

3.4 Initial stress technique

Since the direct iteration based on a variable stiffness needs to form ceaselessly the new stiffness matrix and return ceaselessly to solve, Yang et al. [25, 26] proposed that the initial stress method may be used to solve the elasticity problems with different moduli in tension and compression. At the same time, the employment of isoparametric elements and the judgment of the stress state on Gauss integral point may improve the computational accuracy and speed. The stress-strain relations is

$$\{\sigma\} = [\overline{D}]\{\varepsilon\}, \tag{18}$$

where $[\overline{D}]$ is the symmetric matrix involved the stress state and will degenerate into the classical elastic matrix when moduli of elasticity in tension and compression are the same. Because $[\overline{D}]$ is not known in advance, Eq. (18) may be rewritten as

$$\{\sigma\} = [D_o]\{\varepsilon\} + ([\overline{D}] - [D_o])\{\varepsilon\}, \tag{19}$$

where $[D_o]$ is the given stress-strain matrix in terms of the classical theory. The iterative format may be built as follows

$$\{P^*\}_i = \{P\} - [K]\{d\}_{i+1}, \tag{20}$$

where $\{P^*\}_i = \Sigma\int[B]^T([D(\{d\}_i)] - [D_o])[B]dV\{d\}_i$ and $[K] = \Sigma\int[B]^T[D_o][B]dV$.

Using the initial stress technique only needs to triangularize the stiffness matrix once and avoids the inconvenience introduced by the shear stiffness. However, the choice of the initial stiffness matrix $[D_o]$ has influence on the computational workload and the convergent rate, and $[D_o]$ should be modified timely in the iterative process.

3.5 Smoothing function approach

Yang et al. [27, 28] proposed that the smoothing technique may be used to deal with the bilinear stress-strain relations with different moduli in tension-compression, and built the FEM computational model based on the initial stress technique mentioned above. Because the initial value of p in this method has direct influence on the accuracy and workload of computation, p should be properly adjusted in the iterative process.

4. Analytical method based on simplified model

4.1 Key problem

In the theory presented by Ambartsumyan, although the differential equations of equilibrium and the geometrical equations are the same as those of classical materials theory, with the exception of the physical equations, the governing equation used for solving takes great changes. For example, the consistency equation satisfied by the stress function $\varphi(x,y)$ in the state of plane strain is [11]

$$\nabla^4\varphi(x,y)+\frac{B_3}{b_{11}}\left(\frac{\partial^2 m_1^2\sigma_\beta}{\partial y^2}-2\frac{\partial^2 m_1 m_2\sigma_\beta}{\partial x\partial y}+\frac{\partial^2 m_2^2\sigma_\beta}{\partial x^2}\right)=0\,, \quad (21)$$

where ∇^4 denotes the double Laplacian operator, σ_β is one of the principal stresses, m_1, m_2 are the direction cosine of the principal stress σ_β, B_3, b_{11} are the quantities relevant to the flexibility coefficients. The problem degenerates into the classic expression $\nabla^4\varphi(x,y)=0$ when elasticity moduli in tension and compression are the same, that is, when $B_3=0$. Besides the familiar item $\nabla^4\varphi$ in Eq. (21), there are some nonlinear items involving the principal stress and its direction cosine. To such a complicated equation, it is impossible to get the exact solution in theory, so it may be a feasible way to simplify the mechanical model to obtain the approximate analytical solution.

Ambartsumyan [11] summarized a few analytical solutions of some simple problems, such as the one-dimensional tension problem of bars under deadweight, the one-dimensional problem of gravity bars with two ends fixed, the pure bending problem of beams and round bars, the axisymmetric problem of hollow cylinders (in two states of plane stress and plane strain), the pure bending problem of thin plates, the longitudinal vibrations problem of prismatic bars, and so on. In these problems, there is no shear stress and the principal direction of a point in question coincides with the direction of the normal stress. The form of the physical equation may be determined in advance, and this opens up possibilities for analytical solutions. However, in the state of complex stress, e.g., in a bending beam under a lateral force, shear stresses do exist, so the diagonal tension and diagonal compression of various inclinations and magnitudes are inevitable. In such a case, it is impossible to determine the physical equation in advance, and thus, the solution to the problems is very difficult.

Taking bending beams as studied bodies, Yao and Ye [29-31] proposed a simplified mechanical model. The utilization of this model can extend the solutions from the state of simple stress, which is strictly obtained according to the physical relation on principal direction, into that of complex stress. The numerical results agree well with their analytical solutions. The following is a brief introduction about the simplified model.

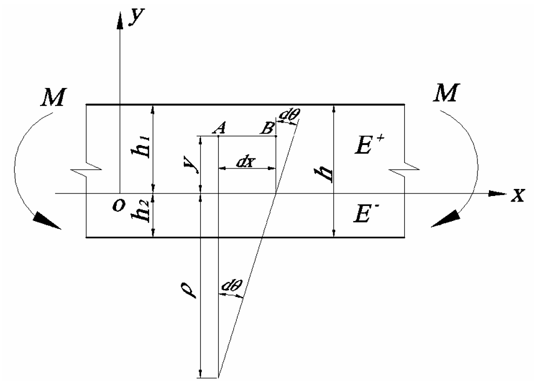

Fig. 2. Bimodular beam under pure bending.

4.2 Simple stress state: pure bending

The bimodular straight beam in pure bending has been solved firstly by Ambartsumyan [11]. Let us consider a rectangular section beam with bimodulus in tension and compression, as shown in Fig. 2, in which M is the bending moment and h is the section height.

The curvature radius of the neutral axis, ρ, may be expressed as

$$\frac{1}{\rho}=\frac{d\theta}{dx}\,, \quad (22)$$

where dx and $d\theta$ are respectively the length and rotation of differential element AB with a distance y from the neutral axis. The relative elongation of AB may be expressed as

$$\varepsilon_x=\frac{(y+\rho)d\theta-\rho d\theta}{dx}=y\frac{d\theta}{dx}=\frac{y}{\rho}\,. \quad (23)$$

From Eq. (23), it may be seen that the longitudinal fiber below the neutral axis is in compression and the longitudinal fiber above the neutral axis is in tension. Due to the simple stress state, the direction of the normal stress in a point exactly coincides with the direction of the principal stress in that point. Therefore, we suppose the section height and Young's modulus in tensile area are h_1 and E^+, respectively, and those in compressive area are h_2 and E^-, respectively. The constitutive relation in tension and compression may be written as

$$\begin{cases}\sigma_x^+=E^+\,y/\rho & 0\le y\le h_1\\ \sigma_x^-=E^-\,y/\rho & -h_2\le y\le 0\end{cases}\,, \quad (24)$$

where σ_x^+,σ_x^- are the normal stresses acted on the section in tension and compression, respectively.

The equilibrium conditions are

$$\int_{-h_2}^{0} \sigma_x^- b dy + \int_{0}^{h_1} \sigma_x^+ b dy = 0 \tag{25}$$

and

$$\int_{-h_2}^{0} \sigma_x^- y b dy + \int_{0}^{h_1} \sigma_x^+ y b dy = M \ , \tag{26}$$

where b is the width of the beam section. Substituting Eq. (24) into (25) and integrating, we obtain

$$E^- h_2^2 = E^+ h_1^2 \ . \tag{27}$$

Considering $h = h_1 + h_2$, we may obtain

$$h_1 = \frac{\sqrt{E^-}}{\sqrt{E^+} + \sqrt{E^-}} h, \ h_2 = \frac{\sqrt{E^+}}{\sqrt{E^+} + \sqrt{E^-}} h \tag{28}$$

Substituting Eq. (24) into (26) and integrating, we obtain

$$\frac{1}{\rho} = \frac{3M}{E^- b h_2^3 + E^+ b h_1^3} \tag{29}$$

If we introduce the flexural stiffness D, such that

$$D = \frac{b}{3}(E^- h_2^3 + E^+ h_1^3) \ , \tag{30}$$

then in the case of small deflection, the differential equation of equilibrium of bimodular beams in pure bending may be expressed as

$$\frac{d^2 y}{dx^2} = -\frac{M}{D} \ . \tag{31}$$

From Eqs. (24) and (31), we may obtain the normal stress

$$\begin{cases} \sigma_x^+ = \dfrac{E^+}{D} My & 0 \le y \le h_1 \\ \sigma_x^- = \dfrac{E^-}{D} My & -h_2 \le y \le 0 \end{cases} . \tag{32}$$

When $E^+ = E^- = E$, all the formulas may be simplified as those in classical beams.

4.3 Yao and Ye's simplified model

Yao and Ye's mechanical model is founded on the following three computational assumptions: (1) The cross section keeps plane in bending. (2) Bounded by the unknown neutral axis, the cross section is divided into the area in tension and the area in compression. Now, the definitions of tension and compression depend on the normal stress acting on the cross section, instead of the principal stress. (3) Shear stress makes no contribution to the position of the neutral axis, i.e. the determination of the neutral axis depends only on the normal stress acted on the cross section and is free of the influences of the shear stress. It is seen that there is some inherent relationship between assumptions (2) and (3) mentioned above because once subarea is assumed, how to determine the position of the neutral axis will become the key problem in a state of complex stress.

4.4 Complex stress state: lateral force bending

Via the simplified model mentioned above, Yao and Ye [30] extended the results from the case of pure bending into the case of lateral force bending. When a bimodular beams is under the action of a lateral force, not only the bending moment M but also the shear force Q acts on the cross section of the beam, that is, not only the normal stress σ_x but also the shear stress τ_{xy} acts on a point in question and the point is in a complex stress state. The diagonal tension and diagonal compression of various inclinations and magnitudes are inevitable. In this case, the neutral axis will not exist if we strictly define tension and compression based on the principal direction. However, from the viewpoint of phenomenalism, a bending beam will always form a deflected shape under the action of a lateral force, and the lower part of the beam is in tension and the higher part is in compression. Therefore, the neutral axis in the case of lateral force bending does exist, like the case of pure bending.

The formulas of the shear stress may be derived by isolating the differential element and considering its equilibrium [30], or by a more convenient method, equivalent section method [32]. In the case of lateral force bending, the formulas of shear stress are

$$\begin{aligned} \tau_{xy}^+ &= \frac{3Q}{2bh}\left[1 - \frac{(\sqrt{E^+} + \sqrt{E^-})^2}{E^- h^2} y^2\right] & 0 \le y \le h_1 \\ \tau_{xy}^- &= \frac{3Q}{2bh}\left[1 - \frac{(\sqrt{E^+} + \sqrt{E^-})^2}{E^+ h^2} y^2\right] & -h_2 \le y \le 0 \end{aligned} \tag{33}$$

when $E^+ = E^- = E$, Eq. (33) may be simplified as that in classical beams.

Yao and Ye analytically investigated a bimodular bending-compression column [29], a bimodular beam subjected to a lateral force [30], a retaining wall with different moduli in tension and compression [31]. Moreover, Yao and Ye compared the analytical solutions of the three problems above with the counterparts in the classical theory and also with the numerical results based on FEM. They also gave some ideas about structural optimization. Their study indicates that the simplified mechanical model is effective for the analytical solution of beams and columns.

4.5 Further research on the existing model

Since the existent derivative pattern is complicate, He et al. [32] proposed that the bimodular problems may be turned into the classical problems via the equivalent section method, and consequently, bending beams subjected to lateral forces and bending-compression column with different moduli can be solved in a simple way. Utilizing the continuity conditions of the stress on the neutral layer, He et al. [33] obtained elasticity solutions of a bimodular simply-supported beam under uniformly-distributed loads. Also, they analyzed the influences introduced by plane section assumption and the bimodularity of materials, respectively. Regarding bimodular bending-compression column as a boundary-value problem, He et al. [34] obtained approximate elasticity solutions of the problem. He et al. [35] analyzed symmetry and antisymmetry of a bimodular elastic structure and concluded that antisymmetry will change due to the bimodular constitutive model.

The flexural rigidity of the structures plays an important role in solving bimodular bending problems. In most cases, the influences introduced by the bimodularity of materials are completely integrated into the flexural stiffness. By the simple substitution to the flexural stiffness, we can readily obtain the solutions of the bimodular problems directly from the known classical problems solutions for a variety of boundary conditions.

It is no doubt that this simplified model is an important assumption because the existent analytical solutions concerning beams and columns are founded on this assumption. Although it has been proved in the range of beam theory, this model still needs an investigation based on 2-D plate theory. It is well-known that there are some similarities between the assumptions in classical beam theory and famous Kirchhoff hypotheses in small deflection theory of thin plates. It could be feasible that using Kirchhoff hypotheses demonstrates this important conclusion about the neutral axis, and at the same time, opens up possibilities for the solving of bimodular plate problems. The relevant work is in progress.

5. Engineering applications

Wu et al. [36] calculated an axisymmetric shell by FEM based on bimodulus; Zhang and Wang [37] analyzed rigid frames by FEM based on bimodulus; Yang et al. [38] presented a high-precision finite element for bimodular shells; Zhang et al. [39] studied influences on mechanical performance of insulators introduced by the bimodularity of materials, and optimized the shape of the suspended insulator [40, 41]; Gong et al. [42] studied expansion of circular cavities with different moduli in tension and compression; Guan et al. [43] analyzed application on rubber water stop in water conservancy strobe by FEM based on large deformation and different moduli; Gao and Liu [44] studied the canopy of plane by theory with different moduli; Gao et al. [45, 46] numerically analyzed bimodular bending plates and thin-shell structures; Gao et al. [47] analyzed structural dynamic characteristics of plates and shells by the theory with different moduli; Liu and Zhao [48] analyzed the stress and the deformation of dams by FEM based on bimodulus; Zhu et al. [49] obtained analytical solution on surrounding rocks in tunnels with different moduli; Luo et al. [50] studied expansion of cylindrical cavities in strain-softening material with different moduli; Yao and Ye [31] studied analytical and numerical solution of the retaining wall based on the theory of different moduli; Zhou et al. [51] derived closed-form analytical solution for beams on elastic foundation with different moduli in tension and compression. These results show that the bimodularity of materials have great influence on the structural stiffness.

For a long time, the analysis and the design of structures have followed the classical elasticity theory based on single modulus. In some cases, however, the neglect of this materials nonlinearity maybe lead to a great computing error because the constitutive model adopted falls short of the actual mechanical performances of materials. This may be the key point which makes structures damage and even failure. Elasticity theory with different moduli in tension and compression deals with many disciplines (e.g. materials, structure, mechanics etc.), and in the long run, it is worthy of some consideration from academic bodies and engineering fields.

6. Conclusions

In this paper, we introduce the basic assumptions of the bimodular materials model proposed by Ambartsumyan and its development, review FEM based on iterative strategy, analytical method based on a simplified mechanical model, and some important engineering applications.

We also point out two challenging problems for the further study: (1) Is tension or compression defined from the point of view of stress or strain? The two macroscopic mechanical models of bimodular materials, the criterion on signs in longitudinal strain of fibers presented by Bert and the criterion on signs in the principal stress presented by Ambartsumyan, essentially deal with the definition on tension and compression, i.e., it is defined either by stress or by strain. For reasons given above, some contrastive researches may be done. (2) This bimodular elasticity theory founded by Ambartsumyan has lacked the experimental results of elasticity coefficients in complex states of stress. Therefore, the proper mechanical model should be rebuilt by extensive experiments.

The work mentioned in this paper will be helpful for predicting the mechanical behaviors of bimodular materials. In particular, these results may be useful to analyze concrete-like materials and fiber-reinforced composite materials that contain cracks and undergoing contact, whose macroscopic constitutive behavior depends on the direction of the macroscopic strain, similarly to the case of the bimodular materials [52-54].

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Nomenclature

E : Young's modulus of elasticity

μ : Poisson's ratio
σ : Normal stress
τ : Shear stress
ε : Normal strain
γ : Shear strain
a,d,B,C : Elasticity coefficients
k_α k_β : Weighted coefficients
$[D]$: Elasticity matrix
$[K]$: Global stiffness matrix
$\{d\}$: Global nodal displacement matrix
$\{P\}$: Global nodal load matrix.
G : Shear modulus of elasticity
η : Convergence factor
ρ : Curvature radius
h : Cross-section height
b : Cross-section width
A : Cross-section area
M : Bending moment
Q : Shear force
D : Flexural Stiffness of beam

Superscript

$+,-$: Quantities related tension and compression, respectively
T : Transpose

Subscripts

x,y,z : General coordinates system
α,β,γ : Principal coordinates system
I : Principal Stress direction

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